HOW TO SOLVE DIFF. EQN. USING INTEGRATING FACTORS

EXAMPLE: CONSIDER THE FOLLOWING DIFF. EQN

$$(3t+10)\frac{dx}{dt} + 3x = 50 - 0.1x(3t+10) \quad \text{WITH} \quad x = 12 \text{ @ } t=0$$

FIRST PUT THE DIFF. EQN INTO THIS GENERAL FORM

$$\frac{dx}{dt} + a(t)x = f(t)$$

SO,

$$(3t+10)\frac{dx}{dt} + 3x + 0.1x(3t+10) = 50$$

$$(3t+10)\frac{dx}{dt} + \left[3 + 0.1(3t+10)\right]x = 50$$

$$\frac{dx}{dt} + \left[\frac{3 + 0.1(3t+10)}{3t+10}\right]x = \frac{50}{3t+10}$$

$$\frac{dx}{dt} + \left[\frac{3}{3t+10} + 0.1\right]x = \frac{50}{3t+10}$$

WITH $a(t) = \frac{3}{3t+10} + 0.1$ ← COULD BE WRITTEN IN OTHER WAYS, BUT THIS WILL TURN OUT TO BE EASIEST

$$f(t) = \frac{50}{3t+10}$$

SECOND, FIND THE INTEGRATING FACTOR $p(t)$ WHERE

$$p(t) = e^{\int a(t)dt}$$ (JUST NEED A FACTOR WITHOUT CONSTANT OF INTEGRATION)

$$a(t) = \frac{3}{3t+10} + 0.1$$

$$\int a(t)dt = \int \frac{3}{3t+10}dt + \int 0.1dt = \frac{3}{3}\ln\left[3t+10\right] + 0.1t$$

$$p(t) = \exp\left[\ln(3t+10) + 0.1t\right]$$

$$= \exp\left[\ln(3t+10)\right]\exp\left[0.1t\right]$$

$$p(t) = (3t+10)e^{0.1t}$$

THIRD, THE SOLUTION WILL BE

$$\frac{d}{dt}\Big[\underbrace{\rho(t)\,x}\Big] = \rho(t)\,f(t)$$

THIS ENTIRE TERM IS DIFFERENTIATED

SO,

$$\frac{d}{dt}\left[(3t+10)e^{0.1t}x\right] = \left[(3t+10)e^{0.1t}\right]\left[\frac{50}{3t+10}\right]$$

$$\frac{d}{dt}\left[(3t+10)e^{0.1t}x\right] = 50e^{0.1t}$$

$$d\left[(3t+10)e^{0.1t}x\right] = 50e^{0.1t}dt$$

INTEGRATE ...

NEED CONSTANT OF INTEGRATE AT THIS POINT.

$$(3t+10)e^{0.1t}x = \frac{50}{0.1}e^{0.1t} + C$$

$$x = \frac{500}{3t+10} + \frac{C}{3t+10}e^{-0.1t}$$

NOTE VERY CAREFULLY HOW CONSTANT OF INTEGRATION HAS BEEN HANDLED.

FOURTH, DETERMINE CONSTANT OF INTEGRATION

$$x = \frac{500}{3t+10} + \frac{c}{3t+10} e^{-0.1t}$$

$x = 12$ WHEN $t = 0$

$$12 = \frac{500}{10} + \frac{c}{10} e^{0}$$

$$(12-50)10 = c \qquad c = -380$$

SO, SOLUTION IS:

$$\boxed{x = \frac{20}{3t+10}\left[25 - 19e^{-0.1t}\right]}$$